MMAT 5340 Assignment #9
Please submit your assignment online on Blackboard
Due at 23:59 p.m. on Tuesday, Apr 9, 2024

1. Suppose we have a box, and N balls in it. Initially, some of these balls are black and the rests are white. Now we repeatedly apply the following procedure:

   - Randomly choose one of the N balls with equal probability and take it out.
   - If the chosen ball is black, we put a white ball into the box.
     If the chosen ball is white, we put a black ball into the box.

   Let X_n be the number of black balls in the box after repeating the above procedure for independently n times. So we know $X = (X_n)_{n\geq 0}$ is a Markov chain with state space $S = \{0, 1, \cdots, N\}$ and the transition matrix P, which is given by

$$P(x,y) = \begin{cases} 1 - \frac{x}{N}, & y = x+1 \\ \frac{x}{N}, & y = x-1 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

   (a) Prove that the Markov chain X is irreducible.
   By the theorem proved in class, there exists a stationary distribution

$$\mu = (\mu(0), \mu(1), \cdots, \mu(N)).$$

   (b) Recall that the stationary distribution μ satisfies $\mu^\top P = \mu^\top$, we obtain $N+1$ linear equations $\mu(n) = \sum_{k=0}^{N} \mu(k)P(k,n)$, for $n = 0, 1, \cdots, N$.
   Please simplify these equations for the transition matrix defined by (1).
   (For example, for $n = 0$, the linear equation is written as $\mu(1)/N = \mu(0)$.)

   (c) Prove that $\mu(2) = \frac{N(N-1)}{2}\mu(0)$ and $\mu(x) = \binom{N}{x}\mu(0)$ for all $x \in S$.

   (d) Compute the stationary distribution μ.

2. Consider the simple random walk $X = (X_n)_{n\geq 0}$ with state space $\mathbb{Z}$ (the set of all integers) and transition matrix P, which is given by

$$P(i,j) = \begin{cases} 1/2, & j = i+1 \text{ or } j = i-1 \\ 0, & \text{otherwise.} \end{cases}$$

   If π is a stationary distribution of X, then

   (a) Prove that $\frac{\pi(x-1)+\pi(x+1)}{2} = \pi(x)$ for all $x \in \mathbb{Z}$.

   (b) Let $u(x) = \pi(x) - \pi(x-1)$ for $x \in \mathbb{Z}$ and prove that $u(x) = C$ for some constant C for any $x \in \mathbb{Z}$.

   (c) Prove that $\pi(x) = ax + b$ for some constant a, b.

(d) Prove that X does not have a stationary distribution.

3. Consider a Markov chain $X = (X_n)_{n\geq 0}$ with state space $\mathbb{N}$ (the set of all nonnegative integers) and transition matrix P, which is given by

$$P(j,k) = \begin{cases} 1, & k = j-1,\ j \geq 1, \\ 0, & k \neq j-1,\ j \geq 1, \\ \nu(k), & k \in \mathbb{N},\ j = 0. \end{cases}$$

where $\nu = \{\nu(n)\}_{n\geq 0}$ is a probability measure on $\mathbb{N}$, i.e. $\nu(n) \geq 0$ for all $n \geq \mathbb{N}$, and $\sum_{n=0}^{\infty} \nu(n) = 1$.

(a) Prove that X is irreducible if and only if $\nu(\{n, n+1, \cdots\}) > 0$ for any $n \in \mathbb{N}$, where $\nu(\{n, n+1, \cdots\}) = \sum_{k=n}^{\infty} \nu(k)$.

(b) Prove that 0 is recurrent.

(c) Prove that the measure defined by $\mu(n) = \nu(\{n, n+1, \cdots\}), n \in \mathbb{N}$ is stationary, i.e. $\mu^\top P = \mu^\top$.